10th lecture

FEYNMAN GRAPHS

in this course, no rigorous derivation (→ QFT)

earlier, we gave some historical arguments

Feynman himself used guesswork & intuition

recall: in lecture 8 we derived Born series →

graphical representation of <u>non</u>relativistic elastic scattering

today: reformulate these results in Feynman's way

→ accept the final rules as reasonable, without proof

before this, some relativistic kinematics ...

Kinematics

- processes of practical importance:
  - decays: 1 parent particle (P^μ) $\longrightarrow$ daughter particles ($p_a'^\mu$)
    seek: differential probability dw to create given set $\{p_a'^\mu\}$
  - collisions: 2 parent particles (p_1^μ, p_2^μ) $\longrightarrow$ debris ($p_a'^\mu$)
    seek: differential cross section $\frac{d\sigma}{d\Omega}$ (kinematical parameters)
- both cases depend on the scattering amplitude
  take this from textbook (previously f, now M_{fi}):
  "the (Lorentz-) invariant amplitude M_{fi} is defined for any process"
- relativistic QFT analog of $\frac{d\sigma}{d\Omega} = |f|^2$ is:
  - for decays: $dw = (2\pi)^4 \delta^{(4)}(\sum_a p_a' - P)\, |M_{fi}|^2 \frac{1}{2E} \overbrace{\prod_a \frac{d^3 p_a'}{(2\pi)^3\, 2\mathcal{E}_a'}}^{\text{LIPS}}$
  - for collisions: $d\sigma = (2\pi)^4 \delta^{(4)}(\sum_a p_a' - p_1 - p_2)\, |M_{fi}|^2 \frac{1}{4I} \prod_a \frac{d^3 p_a'}{(2\pi)^3\, 2\mathcal{E}_a'}$

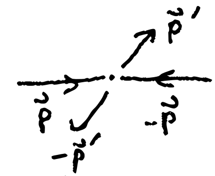
E = energy of decaying particle $\sim (\int dw)^{-1} \sim \tau$ lifetime
I = invariant flux $= \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}$ $\rightsquigarrow$ $\sigma = \int d\sigma$ Lorentz-invariant

$d^4 p_a'\, \delta(p_a'^2 - m_a^2)\, \Theta(\mathcal{E}_a')$

- most interesting case: scattering $2 \to 2$

amplitude depends on 3 kinematic parameters, e.g.

- total energy $E = \varepsilon_1 + \varepsilon_2 = \varepsilon_1' + \varepsilon_2'$
- scattering angles θ, φ in CMS $\longrightarrow$



for scalar particles $\exists$ symmetry of rotation around beam axis

$\rightsquigarrow$ only (E, θ), or $s = (p_1 + p_2)^2$ & $t = (p_3 - p_1)^2$

in CMS: $$d\sigma = \frac{1}{64\pi^2} |\mathcal{M}_{fi}|^2 \frac{|\vec{p}\,'|}{|\vec{p}|\, E^2}\, d\Omega \qquad [\mathcal{M}_{fi} \text{ is dim'less}]$$

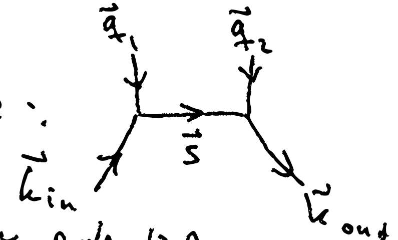
- nonrelativistic limit: $m_2 \gg m_1$, $v_1 \ll c$ $\rightsquigarrow$

heavy particle represents an external potential for slow light one

$$\mathcal{M}_{fi}^{2\to 2} \xrightarrow{\div 2m_2} \mathcal{M}_{fi}^{pot} = 4\pi\, f(\theta, \phi) \quad \text{nonrelativistic}$$

$$\hookrightarrow \quad d\sigma = \frac{1}{16\pi^2} |\mathcal{M}_{fi}^{pot}|^2\, d\Omega$$

Potential scattering



- recall first correction to Born amplitude:

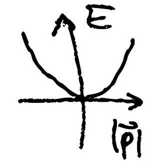
1. $\vec{k}_{in}$ from ∞, scattered with strength $\tilde{V}_{\vec{q}_1}$ by potential, mom. transfer $\Delta\vec{k} = \vec{q}_1$
2. intermediate particle with momentum $\vec{s} = \vec{k}_{in} + \vec{q}_1$
   & energy $E^* = (\vec{k}_{in} + \vec{q}_1)^2/2m \neq E_{in}$
3. energy non-conservation admissible for a moment $\tau \sim \hbar/\Delta E$
4. energy denominator $(E - E^*)^{-1} = \frac{2m}{(\vec{k}_{in} - \vec{s})^2}$
   fund. solution of Schrödinger eqn
5. 2^{nd} scattering, with strength $\tilde{V}_{\vec{q}_2}$ & $\Delta\vec{k} = \vec{q}_2$
   $\rightsquigarrow \vec{k}_{in} + \vec{q}_1 + \vec{q}_2 = \vec{k}_{out}$, $|\vec{k}_{out}| = |\vec{k}_{in}|$ $\rightsquigarrow$ energy conservation
6. fix total $\Delta\vec{k} = \vec{q}_1 + \vec{q}_2 =: \vec{q}$ $\rightsquigarrow \int \frac{d^3q_1}{(2\pi)^3} \int \frac{d^3q_2}{(2\pi)^3} (2\pi)^3 \delta^{(3)}(\vec{q}_1 + \vec{q}_2 - \vec{q}) \times \frac{-m}{2\pi}$

- slightly different interpretation

assume the energy <u>is</u> conserved at each interaction vertex, so the particle keeps its initial = final energy E also in intermediate state (external potential transfers only $\vec{p}$ not E !)

$\rightsquigarrow$ have to give up usual dispersion relation $E = \vec{p}^{\,2}/2m$ for intermediate particle

jargon: "intermediate particle is off mass shell" (or "virtual")



- relativistic version

Schrödinger eq. $\longrightarrow$ KFG equation: $(\Box + m^2)\Psi = $ source

$\downarrow$

relativistic scalar propagator

$$\frac{1}{E - \vec{p}^{\,2}/2m} \longrightarrow \frac{2m}{p_0^2 - \vec{p}^{\,2} - m^2} = \frac{2m}{p^2 - m^2}$$

$$\frac{1}{E - \vec{p}^{\,2}/2m} = \frac{2m}{2mE - \vec{p}^{\,2}} \overset{E \ll m}{\approx} \frac{2m}{(m+E)^2 - \vec{p}^{\,2} - m^2} \quad \left(\frac{2m}{p_0^2 - \vec{p}^{\,2} - m^2} \;\Big\downarrow\; p_0 = m + E\right)$$

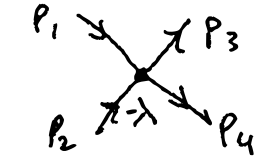
$$G(p) = \frac{1}{p^2 - m^2}$$

[fate of normalization $2m$ to be discussed later...]

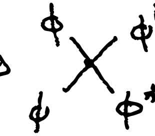
Scalar field theory

replace quadratic Lagrangian + V_{ext} by
a quartic Lagrangian, no V_{ext}
$\longrightarrow$ particles can scatter off one another!

- recall first Feynman diagram from lecture 3 (correct sign)



$$\rightsquigarrow \mathcal{M}^{\text{tree}}_{fi} = -\lambda \iff \mathcal{L}_{int} = -\frac{\lambda}{4}(\phi^*\phi)^2 \rightsquigarrow$$

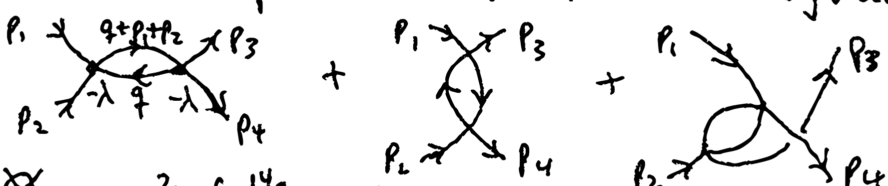


factor of 4 is combinatorial:

e.g.

2 ways each to associate $\begin{cases} p_1, p_2 \to \phi, \phi \\ p_3, p_4 \to \phi^*, \phi^* \end{cases}$

- one-loop correction: 3 diagrams



in each loop: two virtual particles, loop momentum q not fixed $\rightarrow$ must be integrated!

$$\mathcal{M}^{\alpha}_{fi} = -i\lambda^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2-m^2} \frac{1}{(q+p_1+p_2)^2-m^2} \rightsquigarrow \text{is log div!}$$

$\hookrightarrow$ see QFT

Scalar electrodynamics

- Lagrangian density: $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D^\mu \phi)^* (D_\mu \phi)$

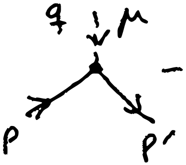
with $D_\mu = \partial_\mu + ieA_\mu$, invariant under $A_\mu \to A_\mu + \partial_\mu \lambda$ & $\phi \to e^{-ie\lambda}\phi$

$$\mathcal{L} = \underset{\text{free}}{\mathcal{L}_2} + \underbrace{\mathcal{L}_3 + \mathcal{L}_4}_{\text{interactions}}, \quad \mathcal{L}_3 = ieA_\mu \left[(\partial^\mu \phi)^* \phi - \phi^* (\partial^\mu \phi) \right]$$

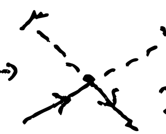
$$\mathcal{L}_4 = e^2 A_\mu A^\mu \phi^* \phi$$

Fourier trsfm.

$$\overleftrightarrow{\partial}^\mu \to i(p+p')^\mu$$

$\mathcal{L}_3 \longrightarrow$ 

$-e(p+p')^\mu$

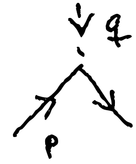
$\mathcal{L}_4 \longrightarrow$ 

$2e^2 \eta^{\mu\nu}$

describes charged scalar particles (e.g. $\pi^\pm$) interacting with an electromagnetic field (i.e. photons γ's)

· consider a scalar particle scattering off a static EM potential

– in leading order:



$q = (0, \vec{q})$ $\quad \mathcal{O}(e)$

$\rightsquigarrow \mathcal{M}^{(1)}_{fi} = -e\,(p+p')^\mu\,\tilde{A}_\mu(\vec{q})$

static electric field: $\tilde{A}_\mu(\vec{q}) = \delta_{\mu 0}\,\tilde{\varphi}(\vec{q})$

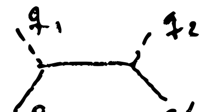
particles slow moving: $(p+p')^0 \approx 2m$

$\Longrightarrow \mathcal{M}^{(1)}_{fi} \approx -2me\,\tilde{\varphi}(\vec{q}) = -2m\,\tilde{V}_{\vec{q}}$

$\rightsquigarrow f^{(1)} = \dfrac{\mathcal{M}^{(1)}_{fi}}{4\pi} \approx -\dfrac{m}{2\pi}\,\tilde{V}_{\vec{q}}$ Born approximation ✓

– in next order:

$+$



$\mathcal{O}(e^2)$

nonrel. limit of  ( does not contribute)

$$\mathcal{M}^{(2)}_{fi} \approx -\int \frac{d^3 q_1}{(2\pi)^3}\, 2me\,\tilde{\varphi}(\vec{q}_1)\, \frac{1}{(p+q_1)^2 - m^2}\, 2me\,\tilde{\varphi}(\underbrace{\vec{q}-\vec{q}_1}_{\vec{q}_2}) \quad \text{with } \vec{q} = \vec{p}\,' - \vec{p} = \vec{q}_1 + \vec{q}_2$$

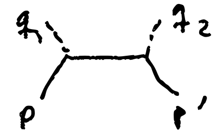
$$\hookrightarrow \frac{1}{2m}\,\frac{1}{E - (\vec{p}+\vec{q}_1)^2/2m + i0}$$

$\rightsquigarrow \mathcal{M}^{(2)}_{fi}/4\pi$ reproduces nonleading Born term $f^{(2)}$ ($\rightarrow$ lecture 8)

- replace static EM lines by real on-shell photons

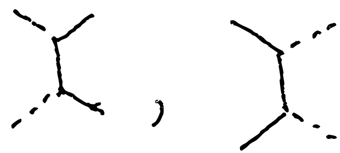
  $\rightsquigarrow$ scattering of photons on scalar charged particles

  e.g. $\gamma\pi^+ \to \gamma\pi^+$ $\qquad q_i^2 = 0$ on-shell



- read sideways (change temporal components of q_i, p, p')

  $\rightsquigarrow$



  recall: incoming negative-energy particle = outgoing positive-energy antiparticle

  describes: $\pi^+\pi^- \to \gamma\gamma$, $\gamma\gamma \to \pi^+\pi^-$

- real (external) photon lines

  vector potential $\tilde{A}_\mu \longrightarrow$ polarization $\varepsilon_\mu(q)$

  transverse polarization: $\varepsilon \cdot q = 0$, $q^2 = 0$

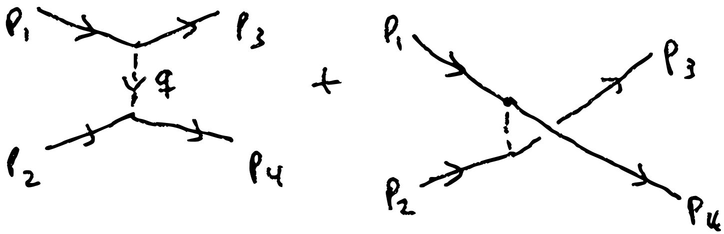
  basis of 2d pol. space

  choose reference system $q = (k, 0, 0, k)$ $\rightsquigarrow \begin{cases} \varepsilon_\mu^{(1)} = (0,1,0,0) \\ \varepsilon_\mu^{(2)} = (0,0,1,0) \end{cases}$

  careful: plane-wave expansion $\rightsquigarrow \varepsilon_\mu(q) \in \mathbb{C}$

  $\rightsquigarrow$ outgoing photons $\sim \varepsilon_\mu^*(q)$

- another process at $\mathcal{O}(e^2)$: $\pi\pi$ scattering



involve virtual photon exchange $\rightsquigarrow$ photon propagator? in Lorentz gauge:

$$D_{\mu\nu}(q) = \frac{\eta_{\mu\nu}}{q^2} \Leftrightarrow \Box A_\mu = \text{source}$$

Sum of these two diagrams $\rightsquigarrow$

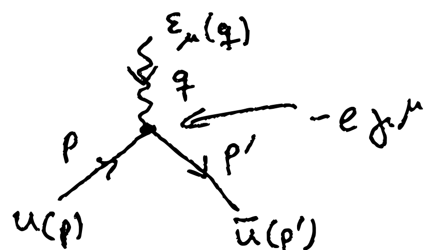
$$\mathcal{M}_{fi}^{\pi\pi\to\pi\pi} = e^2 \left[\frac{(p_1+p_3)\cdot(p_2+p_4)}{(p_1-p_3)^2} + \frac{(p_1+p_4)\cdot(p_2+p_3)}{(p_1-p_4)^2} \right]$$

Spinor electrodynamics

need a theory of electron-photon interactions: spinor QED

- recall: $\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^\mu(\partial_\mu + ieA_\mu)\psi - m\bar{\psi}\psi$

$\rightsquigarrow \mathcal{L}_{int} = -e\bar{\psi}\gamma^\mu A_\mu \psi$, plane waves: $\begin{cases} \psi(x) = \underset{\text{helicities}}{u(p)}\, e^{ip\cdot x} \\ A_\mu(x) = \underset{\text{polarizations}}{\varepsilon_\mu(q)}\, e^{iq\cdot x} \end{cases}$

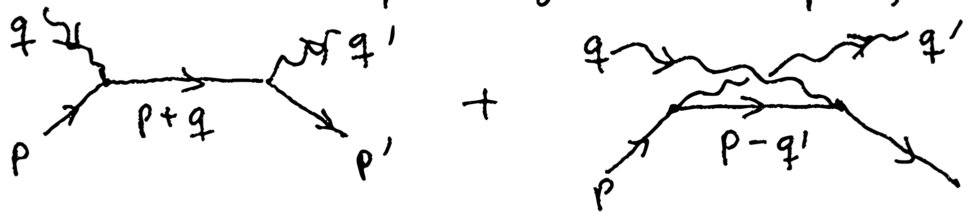


$\rightsquigarrow \mathcal{M}_{fi} = -e\,\bar{u}(p')\gamma^\mu u(p)\,\varepsilon_\mu(q)$

- cross section $\sim |\mathcal{M}_{fi}|^2$
  if electrons are unpolarized, must sum over final & average over initial polarizations

- special cases: [→ lecture 4]
  - external Coulomb field (Ze), nonrel. limit ⟶ Rutherford ✓
  - e^-e^- (Møller) scattering:  similar to scalar QED but γ^μ structure & external u's

- another example: γe^- (Compton) scattering



[pol'ns suppressed]

external photons: $\varepsilon_\mu(q)$ & $\varepsilon^*_\nu(q')$

external electrons: $u(p)$ & $\bar{u}(p')$

vertices: $-e\gamma^\mu$ & $-e\gamma^\nu$

internal electron line: in analogy with photon & scalar:

$(i\not{\partial} - m)\Psi = \text{source} \implies S(p) = \dfrac{1}{\not{p} - m} = \dfrac{\not{p} + m}{p^2 - m^2}$ fermion propagator

all together:

$$\mathcal{M}_{fi}^{\gamma e \to \gamma e} = -e^2 \varepsilon_\mu^{(s)}(q)\, \varepsilon_\nu^{(s')*}(q')\, \bar{u}^{(\sigma')}(p') \left[\gamma^\nu \frac{\not{p} + \not{q} + m}{(p+q)^2 - m^2} \gamma^\mu + \gamma^\mu \frac{\not{p} - \not{q}' + m}{(p-q')^2 - m^2} \gamma^\nu \right] u^{(\sigma)}(p)$$

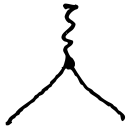
where $\left\{ \begin{array}{l} s = \pm 1,\ s' = \pm 1 \text{ are the photon polarizations} \\ \sigma = \pm 1,\ \sigma' = \pm 1 \text{ are the electron helicities} \end{array} \right\} \rightsquigarrow \mathcal{M}_{fi}^{ss'\sigma\sigma'}(E, \theta, \phi)$

- electroweak theory has massive vector bosons $W^\pm$ & Z:

propagator = Fourier trsfm. of Proca's Green's fctn: $D^M_{\mu\nu}(q) = \dfrac{\eta_{\mu\nu} - q_\mu q_\nu / M^2}{q^2 - M^2}$

Loops

- electron scattered by external magnetic field probes the intrinsic (spin) magnetic moment of e^-



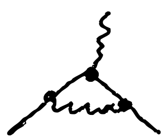
$$\longrightarrow \mu_e = \frac{e\hbar}{2m_e c} = \underset{\text{Landé factor} = 2}{g} \cdot \underbrace{\frac{e\hbar}{2m_e c}}_{\text{Bohr magneton}} \cdot \underset{\text{spin} = 1/2}{s} \approx -9.274 \times 10^{-21} \frac{\text{erg}}{\text{gauss}}$$

Landé factor = 2 ← Dirac equation!

experimental value is ~1‰ larger ...

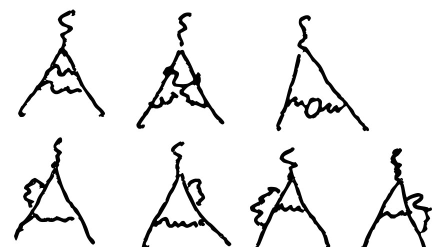
- Schwinger's explanation: need to add QED corrections

leading correction $g = 2\left(1 + \frac{\alpha}{2\pi}\right)$



"one-loop"

2-loop:



- higher-order corrections

2-loop: 7 graphs, 3-loop: 72 graphs } analytically

4-loop: 891 graphs, 5-loops 12672 graphs } numerically

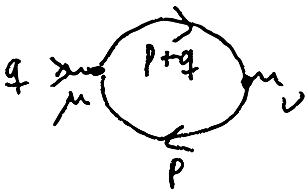
$$\left.\frac{g-2}{2}\right|_{th} = (1159652181\,6 \pm 8) \times 10^{-13}$$

!

$$\left.\frac{g-2}{2}\right|_{exp} = (1159652180\,7 \pm 3) \times 10^{-13}$$

- other QED loops:

charge renormalization



$=: \mathcal{P}^{\mu\nu}(q)$ "photon polarization operator"

$$\mathcal{P}^{\mu\nu}(q) = i e^2 \int \frac{d^4p}{(2\pi)^4} \operatorname{tr}\left\{ \gamma^\mu \frac{1}{\not{p}-m} \gamma^\nu \frac{1}{\not{p}+\not{q}-m} \right\}$$

this is UV-divergent $\sim e^2 \int^\Lambda \frac{d^4p}{p^2} \sim e^2 \Lambda^2$

but doing the cutoff carefully respecting Lorentz & gauge-inv, cancels Λ^2 term $\rightsquigarrow$

$$\mathcal{P}^{\mu\nu}(q^2) = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) \mathcal{P}(q^2)$$

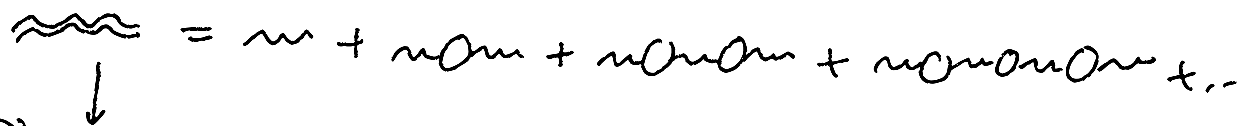
$\mathcal{P}(q^2) \overset{q^2\to 0}{\sim} q^2(\ldots)$

where $\mathcal{P}(q^2) \sim \ln \Lambda$ and $\mathcal{P}(0) = 0$, is transverse:

$\mathcal{P}(q^2)$ can be computed: $\partial_\mu j^\mu = 0 \iff q_\mu \mathcal{P}^{\mu\nu} = 0$

asymptotics: $$\mathcal{P}(q^2, m, \Lambda) \simeq \begin{cases} -\frac{e^2}{12\pi^2} q^2 \ln \Lambda^2/q^2 & \text{for } |q^2| \gg m^2 \\ -\frac{e^2}{12\pi^2} q^2 \ln \Lambda^2/m^2 & \text{for } |q^2| \ll m^2 \end{cases}$$

• full photon propagator:



$$\rightsquigarrow \mathcal{D}(q^2) = \frac{\eta_{\mu\nu}}{q^2} + \frac{\eta_{\mu\rho}}{q^2}\left(\eta^{\rho\sigma} - \frac{q^\rho q^\sigma}{q^2}\right) P(q^2)\,\frac{\eta_{\sigma\nu}}{q^2} + \dots$$

$$\overset{\text{geom. series}}{=} \frac{\eta_{\mu\nu}}{q^2 - P(q^2)} - \frac{q_\mu q_\nu P(q^2)}{q^4\left(q^2 - P(q^2)\right)}$$

2^{nd} term drops out when sandwiched between u's:

$$\bar{u}(p)\gamma^\mu q_\mu u(p') = \bar{u}(p)\gamma^\mu (p-p')_\mu u(p') = 0$$

since $\not{p}' u(p') = m\,u(p')$ & $\bar{u}(p)\not{p} = \bar{u}(p) m$

$\rightsquigarrow$ can always add term $\sim q^\mu q^\nu$ to photon propagator ($\Leftrightarrow$ gauge invariance)

$$\rightsquigarrow \mathcal{D}_{\mu\nu}(q) = \frac{\eta_{\mu\nu}}{q^2 - P(q^2)} \overset{|q^2| \ll m}{\approx} \frac{\eta_{\mu\nu}}{q^2\left(1 + \frac{e^2}{12\pi^2}\ln \Lambda^2/m^2\right)} = \frac{D_{\mu\nu}(q^2)}{1 + \frac{e^2}{12\pi^2}\ln \Lambda^2/m^2}$$

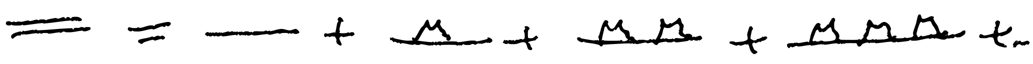
reproduces $e_{phys}^2 = e_0^2 \Big/ \left(1 + \frac{e^2}{12\pi^2}\ln \Lambda^2/m^2\right)$ (1-loop) ← lecture 4

• other QED loops: electron mass renormalization

$=: \Pi(p)$ "electron polarization operator"

is also log. divergent, $\sim \ln \Lambda$

full electron propagator

$\mathcal{S}(p) =$ 

$= S(p) + \dots$ (geometric series)

$= \dfrac{1}{\not{p} - m_{phys}}$ ← phys. mass = pole of full propagator

result: $m_{phys} = m_0 \left(1 + \frac{3e_0^2}{8\pi^2} \ln \Lambda/m_0 \right)$ (1-loop)

can do better: keep all leading log's in higher loops

$\rightsquigarrow m_{phys} = m_0 \left(\frac{e_0^2}{e_{phys}^2}\right)^{9/4} > m_0$ , ...

- the precise spinor QED Feynman rules
  (for the avid practicioner)

1. all vertex expressions read off the Lagrangian are multiplied by i
2. virtual photon line: $-i D_{\mu\nu}(q^2)$
   ingoing photon: $\varepsilon_\mu(q_{in})$, outgoing photon: $\varepsilon^*_\mu(q_{out})$
3. virtual electron line: $i S(p)$
   ingoing electron: $u(p_{in})$, outgoing electron: $\bar{u}(p_{out})$
   ingoing positron: $\bar{u}(-p_{in})$, outgoing positron: $u(-p_{out})$ $(p_0 > 0\,!)$
4. if internal virtual 4-momenta are not determined by kinematics (momentum conservation at each vertex!), one must integrate
   $\prod_j d^4 s^{(j)}/(2\pi)^4$
5. each fermion loop gets an extra minus sign (Grassmann:)
6. the final expression for M_{fi} is to be multiplied by i.
7. there are combinatorial factors from counting contractions leading to same diagrams $\Longrightarrow$ HAPPY COMPUTING ☺